Existence of non-trivial, vacuum, asymptotically simple space-times

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Abstract

We construct non-trivial vacuum space-times with a global $\mathscr{I}^+$. The construction proceeds by proving extension results across compact boundaries for initial data sets, adapting the gluing arguments of Corvino and Schoen. Another application of the extension results is existence of initial data which are exactly Schwarzschild both near infinity and near each of the connected component of the apparent horizon. Finally the construction allows one to add Einstein-Rosen bridges to time-symmetric initial data sets at points satisfying a local parity condition, with the perturbation of the metric localized in an arbitrarily small neighbourhood of the bridge.

1 Introduction

In a recent significant paper [**?**] Corvino has presented a gluing construction for scalar flat metrics, leading to the striking result of existence of non-trivial scalar flat metrics which are exactly Schwarzschild at large distances; extensions of the results in [**?**] have been announced in [**?**]. The method consists in gluing an asymptotically flat metric g with a Schwarzschild metric[1] on an annulus $B(0, 2R_0) \setminus B(0, R_0)$. One shows that if R_0 is large enough, then the gluing can be performed so as to preserve the time-symmetric scalar constraint equation $R(g) = 0$, where $R(g)$ is the Ricci scalar of g.

One would like to use the above method to construct vacuum space-times which admit conformal compactifications at null infinity with a high degree of differentiability *and* with a global $\mathscr{I}^+$. Indeed, metrics which are Schwarzschildian, or Kerrian, near i^0 contain hyperboloidal hypersurfaces of the kind needed in Friedrich's stability theorem [**?**], and if the initial data are close enough to those for Minkowski space-time in an appropriate norm, Friedrich's result yields the required asymptotically simple [**?**] space-time. In Corvino's construction there arises, however, an apparent difficulty related to the fact that if a sequence of data (g_i, K_i) approach the Minkowski space-time, then the gluing radius $R_i = R_0(g_i, K_i)$ above could in principle tend to infinity. This could then lead to hyperboloidal initial data such that the relevant norm for Friedrich's theorem would *fail* to approach zero, barring one from achieving the desired conclusion.

The main object of this letter is to point that the imposition of a parity condition on the initial data sets considered avoids the above mentioned problem. We will actually use a slight variant[2] of the above construction, which produces extensions across a boundary $S(0, R) = \partial B(0, R)$, for any *fixed* radius R, for "small" initial data sets, regardless of whether or not they are originally arising from an asymptotically flat initial data set. As a consequence we can produce an infinite dimensional family of vacuum space-times which are asymptotically simple in Penrose's sense, with a conformal

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[1]Here we mean the metric induced by the Schwarzschild metric on the usual $t = 0$ hypersurface in Schwarzschild space-time; we will make such an abuse of terminology throughout.

[2]We note that one can very well use the original method of [**?**] to construct the asymptotically simple space-times, without using our variation of Corvino's method: Indeed, the arguments given below show that the original argument of [**?**], in the space of parity-symmetric metrics, will lead to a gluing radius which can be chosen to be independent of the metric, for sets of metrics which are bounded in a norm which controls the decay of the metric and of a finite number of its derivatives.

compactification of C^k differentiability class for any finite k; however, the method fails to produce C^∞ compactifications. Recall that the only vacuum, asymptotically simple space-time with a conformal compactification with a C^3 metric known until the examples presented here was Minkowski space-time[3]: this is due to the fact that the differentiability properties of conformal compactifications of the space-times constructed by Christodoulou and Klainerman [?] or Klainerman and Nicolò [?] are only very poorly controlled *a priori*.

As another application of the local extension construction we obtain a family of "many black holes" initial data sets (M, K, g) with the following property: M is a union of a finite number of asymptotically flat regions and a compact set $\mathscr{K}$. The metric is *exactly* Schwarzschild or flat on $M \setminus \mathscr{K}$. Further all the Schwarzschild apparent horizons "$\mathcal{S}_i := \{r_i = 2m_i\}$" are outside of $\mathscr{K}$, so that the metric is exactly Schwarzschildian in a neighbourhood of each of the $\mathcal{S}_i$'s. This provides a family of non-trivial metrics with "isolated horizons", with the geometry being exactly Schwarzschildian in a neighbourhood of the horizons.

Finally, we use the Corvino-Schoen technique to add Einstein-Rosen bridges to manifolds satisfying $R = 0$, assuming in addition a *local* parity condition. The deformation of the metric related to the addition of the bridge preserves the condition $R = 0$ and modifies the metric only in an arbitrarily small neighbourhood of the point at which the bridge is added. This allows one to connect pairs of time-symmetric vacuum initial data without perturbing the metric away from a small neighbourhood of the bridges, or to create wormholes within a given initial data set. (Compare [?, ?], where the conformal method is used for the gluings: this leads *a priori* to conformal deformations of the metrics which are small away from a neighbourhood of the gluing point, but non-zero *throughout* the manifolds being glued.) The construction can be done for general time-symmetric initial data at points at which the Ricci tensor of the three-dimensional[4] metric g does not vanish, *without* any parity conditions, this will be discussed elsewhere [?]. One can similarly add bridges to $R = \text{const} \neq 0$ manifolds, preserving the $R = \text{const}$ condition [?]; this makes use of the relative mass identities of [?, ?]. We expect to be able to extend the bridge-addition technique to vacuum initial data sets with non-vanishing K in a near future.

2 Extensions of initial data sets

We start by considering the *extension problem*, that is, the following question: let us be given a vacuum initial data set (M, K, g), where $\overline{M} = M \cup \partial M$ has a compact boundary ∂M, with the data (K, g) extending smoothly, or in $C^k(\overline{M})$, to the boundary. Does there exist an extension across ∂M of (K, g) which satisfies the constraint equations? In the case where K vanishes and ∂M is *mean outer convex* an affirmative answer can be given by using a method[5] due to Smith and Weinstein [?]. However, it is not clear how that method can be used to produce extensions which are exactly Schwarzschildian outside of a compact set. Further, it is even less clear how to extend this method to accomodate non-trivial extrinsic curvature.

We wish to point out that the results of Corvino and Schoen [?, ?, ?] can be used to obtain alternative extension results, *without* the hypotheses that the boundary is mean outer-convex and that K vanishes. (It further seems that less differentiability of the metric is lost with this method, as compared to the Smith-Weinstein technique; the latter gives C^k extensions of C^{2k+1} metrics, $k \geq 0$.) Thus, assume we have a solution•2.1 $(K, g) \in (C^{k+2} \times C^{k+3})(\overline{M})$, $k \geq 4$, of the vacuum constraints on a manifold $\overline{M}$ with compact boundary. Let M_0 be another manifold such that ∂M_0 is diffeomorphic to ∂M, and let •*2.1:* **ptc:** *k threshold added - Erwann, do you agree?*

[3]The only non-trivial, with a globally regular $\mathscr{I}$, *electro-vacuum* space-times known so far were provided by the Cutler-Wald metrics [?].

[4]Throughout this letter we assume that the space-dimension is three, though several of the observations made here carry over to any dimension $n \geq 3$.

[5]Smith and Weinstein actually assume that ∂M is a two-sphere, but this hypothesis is irrelevant for the discussion here. The special case when the boundary metric is that of a round two sphere has been previously considered by Bartnik [?].

M' be the manifold obtained by gluing M with M_0 across ∂M. Let x be any smooth function defined in a neighbourhood $\mathscr{W}$ of ∂M on M', with $\partial M = \{x = 0\}$, with dx nowhere vanishing on ∂M, and with $x > 0$ on M_0. It is convenient to choose $\mathscr{V} := \mathscr{W} \cap M_0$ to be diffeomorphic to $\partial M \times [0, x_0]$, with x being a coordinate along the $[0, x_0]$ factor.

Suppose, next, that there exists on M_0 a solution (K_0, g_0) of the vacuum constraint equations which is in $(C^{k+2} \times C^{k+3})(\overline{M}_0)$; we emphasize that we do not assume that (K, g) and (K_0, g_0) match across ∂M. Standard techniques allow one to extend (K, g) to a pair $(\widehat{K}, \widehat{g})$ defined on M_0 such that

1. $(\widehat{K}, \widehat{g})$ remains in $C^{k+2} \times C^{k+3}$;

2. $(\widehat{K}, \widehat{g})$ coincides with (K_0, g_0) on $M_0 \setminus \mathscr{V}$;

3. we have

$$\|\widehat{g} - g_0\|_{C^{k+3}(\mathscr{V})} \leq C \sum_{i=0}^{k+3} \|\partial_x^i g|_{\partial M} - \partial_x^i g_0|_{\partial M}\|_{C^{k+3-i}(\partial M)} \ , \tag{2.1}$$

$$\|\widehat{K} - K_0\|_{C^{k+2}(\mathscr{V})} \leq C \sum_{i=0}^{k+2} \|\partial_x^i K|_{\partial M} - \partial_x^i K_0|_{\partial M}\|_{C^{k+2-i}(\partial M)} \ , \tag{2.2}$$

4. and for all $0 \leq i \leq k+1$ it holds that

$$\begin{aligned} &|(\widehat{\nabla})^{(i)} \rho(\widehat{K}, \widehat{g})|_{\widehat{g}} + |(\widehat{\nabla})^{(i)} J(\widehat{K}, \widehat{g})|_{\widehat{g}} \\ &\quad \leq C \left(\|\widehat{g} - g_0\|_{C^{k+3}(\mathscr{V})} + \|\widehat{K} - K_0\|_{C^{k+2}(\mathscr{V})} \right) x^{k+1-i} \ . \end{aligned}$$

Here ρ is the scalar constraint operator and J is the vector constraint operator; the constant C might depend upon $\|\widehat{g} - g_0\|_{L^\infty}$ and $\|\widehat{K} - K_0\|_{L^\infty}$. Our first extension result is obtained under the hypothesis that there are no (Y, N)'s such that $P^*(Y, N) = 0$ on $\mathscr{V}$, where

$$P^*(Y,N) = \left(\begin{array}{l} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l g_{ij} - K_{ij}N + \operatorname{tr} K \ N g_{ij}) \\ \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l \nabla_q Y^l g_{ij} - \Delta N g_{ij} + \nabla_i \nabla_j N \\ \quad + (\nabla^p K_{lp} g_{ij} - \nabla_l K_{ij}) Y^l - N \mathrm{Ric}\,(g)_{ij} + 2N K^l{}_i K_{jl} - 2N (\operatorname{tr} \ K) K_{ij} \end{array} \right) . \tag{2.3}$$

(Nontrivial fields satisfying $P^*(Y, N) = 0$ are called Killing Initial Data (KIDs) [**?**], their existence implies existence of Killing vectors in the associated vacuum space-time.) If (K, g) and its derivatives on ∂M up to appropriate order, as in (2.1)-(2.2), are sufficiently close to (K_0, g_0) and its derivatives on ∂M, then the results of [**?**] allow one to conclude that there exists on $\mathscr{V}$ a vacuum initial data set, close to zero, $(K_0 + \delta K, g_0 + \delta g) \in (C^k \times C^k)(\overline{\mathscr{V}})$, with all derivatives up to order k coinciding with those of (K_0, g_0) on $\{x_0\} \times \partial M$, and with all derivatives up to order k coinciding with those of (K, g) on $\{0\} \times \partial M$.

The above construction has a lot of if's attached, but it does provide new non-trivial extensions in the following, easy to achieve, situation:

1. (K, g) belongs to a one-parameter family of solutions (K_λ, g_λ) of the vacuum constraint equations on M,

2. the vacuum initial data set (K_0, g_0), assumed above to be defined on M_0, is the restriction to M_0 of a vacuum initial data set defined on M', still denoted by (K_0, g_0), with

3. (K_λ, g_λ) converging to $(K_0|_M, g_0|_M)$ as λ tends to zero in $(C^{k+2} \times C^{k+3})(\overline{M})$.

(Replacing M by a neighbourhood of ∂M, it is of course sufficient for all the above to hold in a small neighbourhood of ∂M.) In such a set-up, proceeding as above one obtains an extension for λ small enough when P^* has no kernel on $\mathscr{V}$.

The situation is somewhat more complicated when a kernel is present, or when working with families of metrics near a metric which has a kernel, and this is what one has to face when attempting to construct asymptotically flat metrics with small mass. We consider a situation where $\overline{M}$ is a smooth compact submanifold, with smooth boundary, of $M' = \mathbb{R}^3$, with $K_0 \equiv 0$, and we let g_0 be the Euclidean metric on $\mathbb{R}^3$. The condition that $\overline{M}$ is a submanifold can be made without loss of generality in the following sense: any two dimensional manifold can be embedded into $\mathbb{R}^3$, and so can a tubular neighbourhood thereof (this will of course not be an isometric embedding in general). Replacing $\overline{M}$ by a tubular neighbourhood $(-x_0, 0] \times \partial M$ of ∂M we can thus identify $\overline{M}$ with a subset of $\mathbb{R}^3$. We note that the closure of $\overline{M}$ in $\mathbb{R}^3$ will then have a boundary with two components, $\{-x_0\} \times \partial M$ and $\{0\} \times \partial M$, but we will ignore $\{-x_0\} \times \partial M$ if occurring, and consider only $\{0\} \times \partial M$, which will be assumed to be an exterior boundary of M as seen from infinity. We assume that (K, g) are close to (K_0, g_0):

$$\|g - g_0\|_{C^{k+3}(\overline{M})} + \|K - K_0\|_{C^{k+2}(\overline{M})} < \epsilon \; ; \tag{2.4}$$

and that

$$g(x) = g(-x) \; , \qquad K(x) = -K(-x) \; ;$$

a large family of such initial data can be constructed by the conformal method. Such data will be referred to as *parity-covariant*. Clearly the extensions $(\widehat{K}, \widehat{g})$ can be constructed as to preserve the parity properties, and we will only consider parity-covariant extensions.

For definiteness we consider only the case $\widehat{K} = K_0 = 0$, though the same argument applies (leading to Kerrian extensions) for non-necessarily zero but appropriately small K's. The constructions of [**?**] preserve all symmetries of initial data, so that gluing together "up to kernel" $\widehat{g}$ with standard (non-translated) Schwarzschild metrics g_m with $m \in (-\delta, \delta)$, with $\delta \le \min(1, 1/R)$ small enough, will lead to approximate solutions "up to kernel" $(\widehat{K} + \delta K_m = 0, \widehat{g} + \delta g_m)$ still being parity-covariant. Parity considerations shows that the center of mass of the resulting metrics is zero, so that the only obstruction, in the proof in [**?**], to the requirement that the metric be scalar flat is the vanishing of the integral of $R(\widehat{g} + \delta g_m)$ over $\mathscr{V}$. Let m_0 denote the naively calculated mass of (K, g) using the ADM integral over $S(0, R)$, we have $|m_0| \le C\epsilon$, and by standard identities for the ADM mass integral one finds

$$\int_{[0,x_0]\times\partial M} R(\widehat{g} + \delta g_m) = m - m_0 + O(\epsilon^2) \; .$$

If the reference Schwarzschild metric g_m has mass $m = -m_0 - \epsilon$ we obtain a strictly negative value of the right-hand-side of the last equation if ϵ is small enough. The value $m = m_0 + \epsilon$ leads to a strictly positive value of that right-hand-side; since the difference depends continuously upon m, there exists $m \in (m_0 - \epsilon, m_0 + \epsilon)$ such that the difference is zero.

If K does not vanish one needs to choose a constant $0 \le \lambda < 1$ and restrict consideration to initial data sets satisfying in addition

$$|\vec{p}_0|_\delta \le \lambda m_0 \; , \tag{2.5}$$

where $\vec{p}_0$ is the ADM momentum of (M, K, g). One then concludes, for ϵ small enough, using *e.g.* the Brouwer fixed point theorem, in a manner somewhat similar to that described in [**?**, **?**]. Summarizing, we have proved:

THEOREM 2.1 *Consider parity-covariant vacuum initial data sets $(K, g) \in C^{\ell+3} \times C^{\ell+4}$, $\ell \ge 3$, on a compact smooth submanifold $\overline{M}$ of $\mathbb{R}^3$, suppose that (2.5) holds with some $0 \le \lambda < 1$, and let Ω be any bounded domain containing $\overline{M}$. There exists $\epsilon > 0$ such that if (2.4) holds, then there exists a vacuum $C^\ell \times C^\ell$ extension of (K, g) across that part of ∂M which is homologous to large coordinate spheres in the asymptotically flat region, with the extension being Kerrian outside Ω. If $K = 0$ then* *the condition* (2.5) *is trivially satisfied (thus no* a priori *restriction on the sign of* m_0 *is imposed in that case), and there exists an extension which is Schwarzschildian.*

□

3 Vacuum asymptotically simple space-times

Theorem 2.1 can be used to establish existence of a large class of small-data, vacuum space-times with a global $\mathscr{I}$, the conformally rescaled metric being as differentiable as desired at $\mathscr{I}$ (though perhaps not necessarily smooth); this proceeds as follows: Let M in Theorem 2.1 be a ball $B(R)$ of fixed radius R, we wish to use Friedrich's stability theorem [?] to establish, for data small enough, the existence of a solution, with a global $\mathscr{I}$, of the Cauchy problem with initial data obtained by the extension technique of Theorem 2.1; this proceeds as follows: Let $\mathscr{S}$ be any smooth spacelike hypersurface in Minkowski space-time which coincides with $\{t = 0\}$ for $r \le 2R$, and which coincides with a hyperboloid $(t - T)^2 - r^2 = 1$, for some T, for $r \ge 4R$. Let (K, g) be any initial data constructed as in the proof of Theorem 2.1 by extending from data prescribed on $B(0, R)$, with $(K, g - g_0)$ small in $(H^\ell \times H^\ell)(B(0, 2R))$, for some $\ell \ge 5$, and let $(\mathscr{M}, {}^4g)$ be the maximal globally hyperbolic development thereof. (We note that the differentiability condition will hold if we start with initial data $(K|_{B(0,R)}, g|_{B(0,R)})$ which are in $(C^{\ell+1} \times C^{\ell+2})(B(0, R))$.) If we construct the extension so that the initial data are Kerrian outside $B(0, 2R)$, then the solution will be a Kerr metric ${}^4g_Q$ in the domain of dependence of $\mathbb{R}^3 \setminus B(0, 2R)$. We will identify the hypersurface $\mathscr{S}$ with a hypersurface in $\mathscr{M}$ as follows: first, for $r \ge R$ we introduce in Minkowski space-time Bondi-type coordinates (u, x, θ, φ) by setting $u = t - r$, $x = 1/r$, so that the Minkowski metric η becomes

$$\eta = x^{-2} \left(-du^2 + 2dx\, du + d\Omega^2\right) , \quad d\Omega^2 = d\theta^2 + \sin\theta\, d\varphi^2 , \tag{3.1}$$

with $\mathscr{S} \cap \{x \le 1/(4R)\}$ taking the form

$$\mathscr{S} = \{u = T + \frac{x}{1 + \sqrt{1 + x^2}}\} . \tag{3.2}$$

Suppose, first, that ${}^4g_Q$ is a Kerr metric in its standard form in retarded Eddington-Finkelstein coordinates [?, p. 879] (u, r, θ, φ), with $|m| + |a| \le \delta$ for a δ small enough so that the metric has no (coordinate or else) singularities or horizons throughout the region $r \ge R$, setting again $x = 1/r$ one then has a natural identification between points on $\mathscr{S}$, understood as a hypersurface in Minkowski space-time, with a hypersurface in the exterior region of a Kerr space-time, by using (3.2). In the (u, x, θ, ϕ) coordinate system ${}^4g_Q$ takes the form

$${}^4g_Q = x^{-2} \left(-du^2 + 2dx\, du + d\Omega^2 + O(\delta x)\right) . \tag{3.3}$$

Further, this form of g_Q is preserved under small translations, small boosts, as well as under arbitrary rotations, as long as $|Q| \le \delta$.

If (K, g) are sufficiently close to Minkowski data on $B(R)$ in $(C^6 \times C^7)(\overline{B(0, R)})$, then the initial data $(K_{\mathscr{S}}, g_{\mathscr{S}})$ induced by ${}^4g$ on $\mathscr{S}$ will be close to the Minkowskian data on $\mathscr{S}$ in $(H^5_{\rm loc} \times H^5_{\rm loc})(\bar{\mathscr{S}})$, where the closure $\bar{\mathscr{S}}$ of $\mathscr{S}$ is taken in the conformally completed space-time. This implies that the resulting initial data for Friedrich's conformal equations will be close to those for Minkowski space-time in the norm needed for Friedrich's stability theorem [?], yielding global existence whenever ϵ in Theorem 2.1 is made small enough. (We note that the whole set of initial data needed in Friedrich's theorem requires divisions by the conformal factor Ω, which vanishes on $\mathscr{I}^+$, and such operations could in principle lead to difficulties when working in Sobolev spaces. However, the data are Kerrian near Scri, thus conformally smooth there, and the division by Ω may be safely performed without the need of imposing further restrictions on ℓ.)

A rough estimate shows that if $(K|_{\overline{B(0,R)}}, g|_{\overline{B(0,R)}})$ are in $(C^{\ell+5} \times C^{\ell+6})(\overline{B(0,R)})$, $\ell \geq 1$, then the resulting conformally rescaled metric will be in $C^{\ell}(\overline{\mathscr{M}})$, where $\overline{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}^+$ is the conformally completed space-time. We expect that this can be improved by a closer inspection of Friedrich's system of equations.

For time-symmetric ($K = 0$) initial data the existence of a global $\mathscr{I}$ follows by covariance of Einstein's equations under time-reversal. For general data we can repeat the above argument with the opposite time-orientation and conclude, decreasing ϵ if necessary, that $\mathscr{M}$ will possess conformal completions with a complete $\mathscr{I}^+$ as well as a complete $\mathscr{I}^-$.

4 Initial data with non-connected trapped surfaces ("many black holes initial data")

As a second illustration of the extension technique above we construct time-symmetric initial data for a vacuum space-time with the following properties:

1. There exists a compact set $\mathscr{K}$ such that g is a Schwarzchild metric with some mass parameter m on each connected component of $M \setminus \mathscr{K}$ (in general different m's for different components);

2. let $\mathcal{S}$ denote the usual minimal sphere within the time-symmetric initial data for the Schwarzschild-Kruskal-Szekeres manifold, then M contains $2N{+}1$ such surfaces, with the metric being Schwarzschild in a neighbourhood of each corresponding $\mathcal{S}$.

In fact, (M, g) will be obtained by gluing together $2N + 1$ Schwarzschild metrics with small masses. One can think of (M, g) as an initial data set containing $2N$ black holes. There is, unfortunately, the usual proviso for such initial data, that it is not clear whether or not M contains other minimal surfaces than the Schwarzschild-ones mentioned above; in particular it could be the case that there is a smooth minimal surface that encloses all the other ones, so that the outermost apparent horizon will actually be connected. Further, even if that is not the case, the intersection of the real event horizon with the initial data hypersurface M could turn out to be connected, so that the "many black hole" character of our initial data would actually be an illusion. We believe that the geometry of (M, g) is striking enough to make it interesting even if it turned out to describe a connected black hole after all.

Let us pass now to the description of our construction: choose two strictly positive radii $0 < 4R_1 < R_2 < \infty$, and for $i = 1, \ldots, 2N$ let the points

$$\vec{x}_i \in \Gamma_0(4R_1, R_2) := B(0, R_2) \setminus \overline{B(0, 4R_1)}$$

($B(\vec{a}, R)$ — open coordinate ball centred at $\vec{a}$ of radius R) and the radii r_i be chosen so that the balls $B(\vec{x}_i, 4r_i)$ are pairwise disjoint, all included in $\Gamma_0(4R_1, R_2)$. Set

$$\Omega := \Gamma_0(R_1, R_2) \setminus \left(\cup_i \overline{B(\vec{x}_i, r_i)}\right) . \tag{4.1}$$

We shall further assume that Ω is invariant under the parity map $\vec{x} \to -\vec{x}$. Let

$$\vec{M} = (m, m_0, m_1, \ldots, m_{2N})$$

be a set of numbers satisfying $2m < 2R_1$, $2m_0 < R_1$, $2m_i < r_i$, and let the metric $g_{\vec{M}}$ be constructed as follows:

1. on $\Gamma_0(R_1, 2R_1)$ the metric $g_{\vec{M}}$ is the Schwarzschild metric, centred at 0, with mass m_0;

2. on $\Gamma_0(3R_1, R_2) \setminus \left(\cup_i \overline{B(\vec{x}_i, 4r_i)}\right)$ the metric $g_{\vec{M}}$ is the Schwarzschild metric, centred at 0, with mass m ;

3. on $\Gamma_0(2R_1, 3R_1)$ the metric $g_{\vec{M}}$ interpolates between the two Schwarzschild metrics already defined above using a usual cut-off function;

4. on the annuli $\Gamma_{\vec{x}_i}(r_i, 2r_i) := B(\vec{x}_i, 2r_i) \setminus \overline{B(\vec{x}_i, r_i)}$ the metric is the Schwarzschild metric, centred at r_i, with mass m_i;

5. on the annulus $\Gamma_{\vec{x}_i}(2r_i, 3r_i)$ the metric interpolates between the two metrics already defined above using a usual cut-off function;

6. the masses m_i, $i = 1, \ldots, 2N$ are so chosen, and the gluings are so performed, that the resulting metric is symmetric under the parity map $\vec{x} \to -\vec{x}$.

Clearly $g_{\vec{M}=0}$ is the flat Euclidean metric on Ω, in particular it is vacuum. Obviously the Ricci scalar $R(g_{\vec{M}})$ of $g_{\vec{M}}$ is symmetric under the parity map and satisfies

$$|R(g_{\vec{M}})| \leq C|\vec{M}| \ .$$

Suppose that

$$|\vec{M}| \leq \delta \ ; \tag{4.2}$$

The results in [?] show that for any k there exists δ_k small enough such that if (4.2) holds with $\delta = \delta_k$, then there exists a C^k metric $\hat{g}_{\vec{M}}$ on Ω which agrees with $g_{\vec{M}}$ at $\partial\Omega$, together with derivatives up to order k, and which is Ricci scalar flat except for the projection on the kernel of P^* (with $Y = 0$, since we are assuming that $K = 0$). Parity considerations as in Section 2 show that the obstruction is the non-vanishing of the integral

$$\int_\Omega R(\hat{g}_{\vec{M}}) \quad = \quad m - \sum_{i=0}^{2N} m_i + O(\delta^2) \ . \tag{4.3}$$

Fix any set of m_i's, $i = 0, \ldots, 2N$, satisfying

$$\sum_{i=0}^{2N} |m_i| \leq \delta/4 \ .$$

If δ is small enough the right-hand-side with $m = \delta/2$ will be strictly positive; it will be strictly negative with $m = -\delta/2$, by continuity there exists m such that $\hat{g}_{\vec{M}}$ will be Ricci scalar flat.

The mass of the solution obtained above, as seen from the end $r \geq R_2$, might be very small. One can now make a usual rescaling $m \to \lambda m$, $r \to \lambda r$, $g_{\vec{m}} \to \lambda^{-2} g_{\vec{m}}$, to obtain any value of the final mass m.

We emphasize that the mass parameters m_i and m_0 are only restricted in absolute value, so solutions with some of the m_i's negative or zero, and/or m_0 negative or zero, and m negative, can be constructed. For instance, a zero value of m_i will correspond to metrics which can be C^k matched to a flat metric on $B(\vec{x}_i, r_i)$. One can actually also obtain $m = 0$: it suffices to repeat the above argument with prescribed values $m = 0$ and m_i, $i = 1, \ldots, 2N$, adjusting m_0 rather than m. This leads to a non-trivial Ricci-scalar flat metric which is flat on an exterior region $\mathbb{R}^3 \setminus B(0, R)$. (Clearly $m = 0$ implies that at least one of the m_i's, $i \geq 0$, is negative, unless they all vanish.)

5 Adding Einstein-Rosen bridges

In the manifold of the last section one can identify points on pairs of some of those resulting minimal surfaces which have the same radius, obtaining — instead of asymptotically flat regions — wormholes connecting neighbourhoods of the corresponding points $\vec{x}_i$. This is a special case of a more general construction, which proceeds as follows: Let $p_0 \in M$ and suppose that there exists a coordinate neighbourhood of $p_0 = \{x^i = 0\}$ in which the metric satisfies the parity condition $g(x) = g(-x)$. As will be seen shortly, it is convenient to replace the coordinates by harmonic ones, with $g_{ij}(0) = \delta_{ij}$, $\partial_k g_{ij}(0) = 0$ — this can be done without violating the parity condition. In a manner essentially identical to that described in the previous section, on a coordinate annulus $\Gamma_0(\epsilon, 4\epsilon)$ we can make a deformation of the original metric g to a metric which will be a Schwarzschild one, with small mass m_ϵ, for $r < \epsilon$. In order to work on a fixed set it is convenient to make a rescaling from the annulus $\Gamma_0(\epsilon, 4\epsilon)$ to $\Gamma_0(1, 4)$. There exists ϵ_0 such that for $0 < \epsilon < \epsilon_0$ the Corvino technique produces a deformation which is "Ricci-scalar flat modulo kernel". Parity guarantees that the only obstruction to existence of a solution is the integral $\int_{\Gamma_0(1,4)} R$, and, arguing as before, for ϵ small enough we can choose the mass $m_\epsilon \approx m(\epsilon)/\epsilon$ of the Schwarzschild metric so that the final metric has vanishing scalar curvature. Here $m(\epsilon)$ is the ADM mass integral, calculated at $\partial B(0, 4)$, of the scaled metric $g_\epsilon(x) := g(\epsilon x)$. (The factor $1/\epsilon$ in m_ϵ arises from scaling back to the original annulus). In order to produce an Einstein-Rosen bridge we have to make sure that m_ϵ is positive. Since the scaled metric differs from the flat one by terms of order ϵ^2 or higher we have $|m(\epsilon)| \leq C\epsilon^2$. By the harmonic-coordinate calculations of Bartnik [**?**, Eq. (5.15)] the mass $m(\epsilon)$ is strictly positive for all sufficiently small ϵ's unless there exists a ball around the origin in which the metric is exactly flat. (Here one might wish to redefine the ADM mass integrand by adding to it the supplementary, higher order, terms arising from an integration by parts as in the proof of [**?**, Theorem 5.2]; this will not affect the previous arguments.)•5.1 Assuming that this last possibility does not occur we thus obtain a family of deformed metrics, parameterized by a small parameter ϵ, with $m_\epsilon \to_{\epsilon\to 0} 0$. We can do this construction at two points $p_a \in M_a$, $a = 1, 2$ at which the parity condition holds, with free sufficiently small parameters ϵ_a, $a = 1, 2$ at each of the points. As the masses of the added Schwarzschild metrics tend to zero with ϵ_a tending to zero, those masses can be matched by choosing the ϵ_a's appropriately. One can then glue $M_1 \setminus B(p_1, \delta_1)$ and $M_2 \setminus B(p_2, \delta_2)$ at the minimal neck of the Einstein-Rosen bridge, obtaining a scalar flat manifold with topology $M_1 \# M_2$, where $\#$ is the connected sum operation. For p_a's belonging to the same manifold M the above construction leads to a family of wormholes connecting $M \setminus (B(p_1, 4\epsilon_1) \cup B(p_2, 4\epsilon_2))$ with itself.

•*5.1:* **ptc:** *sentence added*

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